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# Borcherds symmetries in M-theory

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Abstract: It is well known but rather mysterious that root spaces of the $E_k$ Lie groups appear in the second integral cohomology of regular, complex, compact, del Pezzo surfaces. The corresponding groups act on the scalar fields (0-forms) of toroidal compactifications of M theory. Their Borel subgroups are actually subgroups of supergroups of finite dimension over the Grassmann algebra of differential forms on spacetime that have been shown to preserve the self-duality equation obeyed by all bosonic form-fields of the theory. We show here that the corresponding duality superalgebras are nothing but Borcherds superalgebras truncated by the above choice of Grassmann coefficients. The full Borcherds' root lattices are the second integral cohomology of the del Pezzo surfaces. Our choice of simple roots uses the anti-canonical form and its known orthogonal complement. Another result is the determination of del Pezzo surfaces associated to other string and field theory models. Dimensional reduction on $T^k$ corresponds to blow-up of $k$ points in general position with respect to each other. All theories of the Magic triangle that reduce to the $E_n$ sigma model in three dimensions correspond to singular del Pezzo surfaces with $A_{8-n}$ (normal) singularity at a point. The case of type-I and heterotic theories if one drops their gauge sector corresponds to non-normal (singular along a curve) del Pezzo's. We comment on previous encounters with Borcherds algebras at the end of the paper.

Keywords: M-Theory, String Duality, Global Symmetries, Differential and Algebraic Geometry.

*Dedicated to Prof. S. Hawking on his 60th birthday.*

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## Contents

## 1. Introduction

U-duality groups are discrete, apparently arithmetic, groups of symmetries of quantum string models in various compactifications. They act on the scalar fields which are themselves coupled to instantonic sources. In the classical supergravity (low energy) limit the symmetry group $G$ is the real form of this discrete group $G_{\mathbb{Z}}$. More precisely in the supergravity limit the moduli space of vacua for the scalar fields is the symmetric space $G/KG \equiv S$ where $KG$ is the maximal compact subgroup of $G$ and $S$ the solvable subgroup given by the Iwasawa decomposition theorem. In the triangular gauge one is left with a solvable algebra $S$ of symmetries. This algebra has been extended to a superalgebra in 1998 to include the other p-form fields which are coupled to $(p-1)$-branes. Recently Iqbal, Neitzke and Vafa [1] have shown that 1/2 BPS brane types of M-theory compactified on (rectangular) tori almost exactly correspond to spherical (i.e. genus 0) generators in the second cohomology of some associated del Pezzo surfaces. The integral cohomology contains in the classical fashion the root lattice of the U-duality algebra $G$.

It turns out that the full cohomology of these surfaces spans the root lattice of a Borcherds superalgebra. Actually it is a Borcherds algebra for 10d IIB theory, specifically it is the rank 2 toy algebra studied by R. Slansky in [2]; the corresponding Cartan matrix is precisely given by the opposite of the intersection form on the surface. More generally in the presence of fermionic simple roots of length squared equal to one it turns out one must consider them as isotropic roots in order to fit into the Borcherds framework keeping the rest of the intersection form unchanged. The main effects of this choice are to prevent twice the odd roots to be roots and to preserve the symmetry under the (real roots') Weyl group. The degree truncation of [1] corresponds exactly to the possible values of the degrees of differential forms that appear in the finite dimensional (super)algebra of symmetries of the supergravity approach [3]. One Cartan generator can be eliminated as well: it does not couple to any propagating field potential and corresponds to the anticanonical class (see definition below).

Section 2 is devoted to mathematical prerequisites in a condensed but hopefully useful arrangement. In sections 3 and 4 we analyze M-theory as well as its toroidal compactifications from the point of view of the (dual) del Pezzo (projective) surfaces. Serre duality implements on the projective surfaces the Hodge self-duality in spacetime, it is not a symmetry of the full Borcherds algebra but only of the physical equations of motion. Section 5 is the extension to type-I or heterotic theories and this requires nonnormal (ie singular along a curve) surfaces. This may require the introduction of a normalizing surface as an auxiliary space, but we shall work on the singular variety as much as possible. Their Borcherds algebras are constructed in two ways: firstly following the regular case, namely starting from the known U-duality algebra and adding the p-form fields, secondly by using the projection from type II to type I as the fixed point set of an involution which works for the U-duality algebras [4] and also for the full Borcherds superalgebras. Finally we prove that the set of subtheories called the Magic triangle (with simply laced split duality groups) also admit dual surfaces but now with normal singularities. We explain the symmetry of the triangle with respect to the diagonal by using the correspondence between

$A_p$ systems of divisors and the toroidal chain of compactifications. The structure can in fact be generalised to many other triangles. We also discuss some alternative theories that arise in the systematic combination of regular and singular contractions. In the concluding section we recall other occurrences of Borcherds algebras and present a program for further work.

## 2. Geometrical prerequisites

All the surfaces involved in Physics via this paper will be complex, algebraic and projective algebraic surfaces, i.e. projective varieties of complex dimension two. An algebraic (projective) variety is a complex manifold which can be described as the zero locus of some homogeneous polynomials in the $(n+1)$ coordinates of points of $\mathbb{P}^n$ (for some $n$), in particular it is compact, but possibly singular. One may think of these surfaces as some kind of twistor spaces and then of spacetimes as derived objects. In fact they became relevant for string theory in [1] and we refer to that paper for introductory material and a partial dictionary of physical objects. The connections between $E_n$ root lattices, del Pezzo surfaces and maximal supergravity are of course much older. For the sake of brevity we shall for the most part restrict our general considerations to regular surfaces in this section, the details and subtleties of the general case are only alluded to here and left for later publications.

### 2.1 Divisors and their classes

On a compact algebraic variety $X$, a *Weil divisor* [5, 6] is a finite formal linear combination $\sum a_i V_i$ of irreducible analytic subvarieties of complex codimension one $V_i$ with integral coefficients $a_i$. If all $a_i$'s are positive, the divisor is called effective. If $f$ is a meromorphic function on $X$, its zeroes define codimension one subvarieties $Z_i$ (with zeroes of respective order $a_i$'s) and poles define subvarieties $P_j$ (of order $b_j$). One then associates to $f$ a divisor $(f) = \sum a_i Z_i - \sum b_j P_j$. Such divisors are called *principal*. Two divisors are *linearly equivalent* if their difference is principal, and one can mod out the group of Weil divisors $\mathrm{Div}(X)$ by principal (Weil) divisors to obtain the *divisor class group* of $X$: $Cl(X)$.

One can also define *Cartier divisors* as global sections of the quotient (multiplicative) sheaf $\mathcal{M}^*/\mathcal{O}^*$, where $\mathcal{M}^*$ is the sheaf of meromorphic functions on $X$ not identically zero and $\mathcal{O}^*$ the sheaf of nonzero holomorphic functions on $X$. (Local sections of a sheaf are nothing but the defining functions or objects over an open set of the base.)

Meromorphic functions trivially define Cartier divisors which are still called *principal* and induce linear equivalence. In fact, Cartier divisors on $X$ are those Weil divisors which are locally principal (with the intuitive meaning of locally). If $X$ is smooth all Weil divisors are Cartier. So the notion of Cartier divisor becomes important in the singular case.

Given a Cartier divisor $D$ on $X$, one can construct a line bundle on $X$ in the following way. As $D$ is locally principal it can be represented by meromorphic functions $f_i$ on an open cover $\{U_i\}$ of $X$, with $f_i/f_j$ holomorphic and non-vanishing on $U_i \cap U_j$, one may then take $f_i/f_j$ as the transition functions defining the associated line bundle on $X$. Linearly equivalent Cartier divisors give isomorphic line bundles, and one gets a morphism from the Cartier divisor class group ($CaCl(X)$) into the group of isomorphism classes of line bundles on $X$ (which is called the *Picard group* $\mathrm{Pic}(X)$). In fact this is an isomorphism when $X$ is projective as we have assumed here [7].

### 2.2 Intersection on algebraic surfaces

On a *normal* surface (which means that the singular locus on $X$ has codimension strictly larger than one) $X$, divisors are generated by curves, and one can define an intersection number between divisors. If we restrict ourselves at first to smooth surfaces, for smooth curves intersecting transversally the intersection number is simply the (nonnegative) number of intersection points and this extends to other divisors. More precisely, there exists a unique symmetric additive pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ which depends only on the linear equivalence classes and reduces to the number of intersection points for nonsingular curves meeting transversally. Even for nonnormal projective surfaces the pairing is still defined and integer on Cartier divisors [8, prop. 1.8].

If $C$ and $D$ are two curves on $X$ and if one writes $c_1(Z)$ for the first Chern class of the line bundle associated to a curve $Z$, this pairing can be expressed as

$$C.D = \int_D c_1(C) = \int_C c_1(D) = \int_X c_1(C) \cup c_1(D)\,. \tag{2.1}$$

On a smooth compact rational surface, the intersection matrix is unimodular, this reflects Poincaré duality. In the singular case the intersection matrix may have a Kernel and dividing out by the Kernel is called numerical equivalence. The nondegenerate part of the signature is recalled at the beginning of section 3.

### 2.3 Ample divisors, projective embeddings and degrees

A line bundle $\mathcal{L}$ on an algebraic surface $X$ is called *very ample* if, for some n, it has $n+1$ linearly independent global sections that can be used to define an embedding of $X$ in $\mathbb{P}^n$ and $\mathcal{L}$ is then the pull-back to $X$ of the tautological bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$. Moreover, any projective embedding of $X$ is given in that way. A line bundle is *ample* if it has a finite (positive) tensorial power which is very ample. We shall say that a divisor is ample or very ample if its associated line bundle is.

Given an ample divisor $H$ on a regular surface $X$, one defines the ($H$-)degree of any divisor $C$ as $H.C$. This degree $d_H$ gives a morphism from the $\mathbb{Z}$-module $\mathrm{Pic}(X)$ to $\mathbb{Z}$. If $H$ is very ample its degree $H.H$ is equal to the (algebraic) degree of the corresponding projective embedding. The *Nakai-Moishezon criterion* asserts that a divisor $H$ is ample if and only if $H^2 (= H.H) > 0$ and $H.C > 0$ for any irreducible curve $C$. We shall consider singular normal projective surfaces but define the degree of Cartier divisors by requiring the corresponding ample divisor itself to be Cartier.

### 2.4 Canonical class $K_X$

For a normal complex variety $X$ of dimension $n$, the $n$-th tensorial power of the cotangent bundle is a line bundle, and one can therefore associate to it a divisor class $K_X$, the *canonical class*. Its dual the tangent bundle is associated to the *anticanonical* class $-K_X$.

An important relation which holds for any nonsingular curve $C$ on a smooth algebraic projective surface $X$ is known as the *Adjunction Formula*:

$$(K_X + C).C = -2 + 2g(C)\,, \tag{2.2}$$

where $g(C)$ is the arithmetical genus of $C$ it coincides with the geometrical (of Riemann surface theory) genus of $C$ for a regular irreducible curve, sometimes we shall just call it the genus. Note that the relation (2.2) holds also for all normal singular surfaces considered in this work, which have only *Du Val* singularities (this term will be explained in section 6.2).

The *virtual genus* of any divisor $C$ (effective or not) can be defined with this formula as

$$g_v(C) = 1 + \frac{(K_X + C).C}{2}\,. \tag{2.3}$$

It obviously reduces to the geometrical genus for any nonsingular irreducible curve on a smooth surface. One notes that for any rational (i.e. of genus 0) divisor $C$ its Serre dual: $-K_X - C$, is also rational.

### 2.5 Blowing up points

For $\mathbb{C}^2$, one defines the *blow up* of the origin as the surface $X$ defined by $\{(z,l) \in \mathbb{C}^2 \times \mathbb{P}^1 | \forall i,j \ z_i l_j = z_j l_i\}$. In other words, the origin is replaced with the $\mathbb{P}^1$ of all directions of lines passing through it. As this procedure is local, one can blow up in the same way any smooth point P of an algebraic surface $X$. If Y is the surface thus obtained, we get a projection morphism $\pi : Y \to X$ such that $\pi^{-1}(P) \simeq \mathbb{P}^1$ and $Y \setminus \pi^{-1}(P) \simeq X \setminus P$.

Denoting by $E$ the divisor class of the *exceptional curve* $\pi^{-1}(P)$, one can prove that $E$ has self-intersection $E^2 = -1$ and is perpendicular to the pull back of any divisor of $X$ not passing through P. We also have the relation between the canonical classes $K_Y = \pi^* K_X + E$ from which follows $K_Y^2 = K_X^2 - 1$.

Conversely, if $E$ is a smooth curve isomorphic to $\mathbb{P}^1$ and of self-intersection $-1$, the *Castelnuovo-Enriques criterion* asserts that Y can be blown down to a surface $X$ such that $E$ is contracted to a smooth point P and Y is precisely the blow up of $X$ at the point P as described above.

More generally if a curve $E$, still isomorphic to $\mathbb{P}^1$, is of self-intersection $-n$, it can be contracted to a point if and only if $n$ is positive. If $n$ is larger than 1, we get a singular point. The relation between canonical divisors generalizes to $K_Y = \pi^* K_X + (2-n)E$. (In what follows, we often omit $\pi^*$ when there is no ambiguity.)

## 3. From del Pezzo surfaces to Borcherds superalgebras

A *(generalized) del Pezzo surface* $X$ is by definition a connected surface that is possibly singular but *Gorenstein* (i.e. its anticanonical class $-K$ is a Cartier divisor) and is such that $-K$ is ample (hence $X$ is projective).

We consider the vector space $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ over $\mathbb{R}$ of dimension $n$ with the induced bilinear form given by the intersection matrix. According to the *Hodge Index Theorem*, the signature of the bilinear form modulo its Kernel is $(1, n-1)$. $-K$ is a positive (or timelike) direction as it is ample by definition.

Here comes our basic rule, we want to extend the structure of the instantons in the orthogonal hyperplane to $K$ to other rational curves, this suggests the following procedure. Let us assume that the set of *positive roots* contains the rational (i.e. of vanishing virtual genus) divisor classes of nonnegative degree and also (except in the case of M-theory) the anticanonical class $-K$, which has also a positive degree by the del Pezzo defining property. We must now choose among the set of positive roots a basis of *simple roots* $\alpha_i$ such that any positive root can be written as a linear combination of simple roots with positive, integral coefficients. In fact, we observe that these $\alpha_i$'s generate the full Picard lattice of the del Pezzo surface X. $-K$ will turn out to have some multiplicity equal to the rank of the root space minus one (this will be related by Hodge duality to the fact that we introduce as many scalar fields as the number of generators of a codimension one Cartan subspace). In degree zero one is handling a usual Lie root lattice and one can use the classical positive root decomposition. We note that all simple roots verify $\alpha_i^2 \geq -2$. For rational divisors it follows from the adjunction formula and the positiveness of their degrees. One sees that the same bound holds also for $-K$.

We can now extract minus the *intersection matrix* $A_{ij} = -\alpha_i \cdot \alpha_j$ and a $\mathbb{Z}$ (resp. $\mathbb{Z}_2$)-graduation grad given by $\mathrm{grad}(\alpha_i) = -K \cdot \alpha_i \equiv d_{-K}(\alpha_i)$ (resp. mod 2) from the cohomology. By the adjunction formula the $\mathbb{Z}_2$-grading is precisely the squared norm of the divisor mod 2. This $\mathbb{Z}_2$ parity of the degree will correspond to the parity of the degree of the field potential in the supergravity theory. In other words the fermionic character of the roots is dictated by the cohomology multiplicative structure.

However it turns out that whenever a fermionic divisor (root) of square $-1$ appears it should be viewed as an SL(1|1) superroot rather than an OSp(1|2) superroot, i.e. it should have zero Cartan(-Killing) norm. Furthermore the theory of Borcherds superalgebras is best developped for the case of null simple fermionic roots. We must assume the generating fermionic simple root(s) (there are at most two of them in a diagram) to be isotropic but we keep its (their) intersection values with the other generators and the rest of the intersection form as given from cohomology. The corresponding modified matrix will be our *Cartan matrix* $a_{ij}$. This matrix $a_{ij}$ satisfies the following three properties and thus defines a *Borcherds superalgebra* (or *Generalized Kac-Moody* superalgebra) [9, 10], in fact the last condition will be vacuous.

$$a_{ij} \leq 0 \quad \text{if} \quad i \neq j \tag{3.1}$$

$$2a_{ij}/a_{ii} \in \mathbb{Z} \quad \text{if} \quad a_{ii} > 0\,, \qquad \text{for} \quad \mathrm{grad}(\alpha_i) = 0 \tag{3.2}$$

$$a_{ij}/a_{ii} \in \mathbb{Z} \quad \text{if} \quad a_{ii} > 0\,, \qquad \text{for} \quad \mathrm{grad}(\alpha_i) = 1 \tag{3.3}$$

We do not know of any conceptual proof of that fact but as far as (3.2) is concerned, it expresses the integrality of the corresponding $a_{ij}$'s because $-\alpha_i^2 \leq 2$ here. A similar correspondence between the Picard lattices of $K3$ surfaces and generalized Kac-Moody superalgebras without odd "real" (i.e. of strictly positive $a_{ii}$) simple roots has been considered in [11].



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The Borcherds superalgebra associated to the matrix $a_{ij}$ has a Cartan subalgebra $H$, with basis $\{h_{\alpha_i}\}$, it is by definition the Lie superalgebra $G$ generated by $H$ and by the elements $e_{\alpha_i}$, $f_{\alpha_i}$ satisfying here the following elementary relations and their consequences:

$$[e_{\alpha_i}, f_{\alpha_j}] = \delta_{ij} h_{\alpha_i} \tag{3.4}$$
$$[h_{\alpha_j}, e_{\alpha_i}] = a_{ij} e_{\alpha_i}\,, \qquad [h_{\alpha_j}, f_{\alpha_i}] = -a_{ij} f_{\alpha_i} \tag{3.5}$$
$$[h_{\alpha_i}, h_{\alpha_j}] = 0 \tag{3.6}$$
$$ad(e_{\alpha_i})^{1-2\frac{a_{ij}}{a_{ii}}} e_{\alpha_j} = 0 = (ad(f_{\alpha_i}))^{1-2\frac{a_{ij}}{a_{ii}}} f_{\alpha_j} \quad \text{if} \quad a_{ii} > 0 \tag{3.7}$$
$$[e_{\alpha_i}, e_{\alpha_j}] = 0 = [f_{\alpha_i}, f_{\alpha_j}] \quad \text{if} \quad a_{ij} = 0 \tag{3.8}$$

In the following sections, we will compute the Borcherds superalgebra for each del Pezzo surface $X$. Then, by truncating the set of positive superroots, we will recover a superalgebra that preserves the classical supergravity equations of motion. The roots orthogonal to the canonical class are related to the usual duality groups even in the singular surface cases we shall encounter. The divisors with vanishing virtual genus will correspond to $\frac{1}{2}$-BPS states of these theories.

## 4. Smooth del Pezzo's

The smooth del Pezzo surfaces are classified according to what turns out to be the uncompactified spacetime dimension $d = K \cdot K + 2$ [5, 12]:

| | | |
|---|---|---|
| (i) | if $d = 11$ | $X \simeq \mathbb{P}^2$ |
| (ii) | if $d = 10$ | $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $X \simeq \mathbb{B}_1$ |
| (iii) | if $3 \leq d = 11 - k \leq 9$ | $X \simeq \mathbb{B}_k$ |

where $\mathbb{B}_k$ is the surface obtained by blowing up $k \leq 8$ points in general position on $\mathbb{P}^2$.

### 4.1 M theory

We will show that this theory corresponds to the regular surface $\mathbb{P}^2$. First, we note that $\mathrm{Pic}(\mathbb{P}^2) \simeq \mathbb{Z}$ and we can take the class $H$ of a line as generator. Since any two lines are linearly equivalent, and since two distinct lines meet in one point, we have $H.H = 1$. This determines the intersection pairing on $\mathbb{P}^2$ by linearity. The anticanonical class is $-K = 3H$. Using the Nakai-Moishezon criterion, we can easily prove that $-K$ is ample and we can define the corresponding degree $d_{-K}$. The divisors D with a vanishing virtual genus and a degree $d_{-K}(D)$, such as $0 \leq d_{-K}(D) \leq 9$, are given by $D_{M2} = H$ and $D_{M5} = 2H$.

Here we may generalize [1] and remark that the (anticanonical) degree is actually equal to the degree of the potential differential form coupled to the corresponding brane i.e. its spacetime dimension. In [1] this was noticed as a coincidence, here we deduce this fact from the generalized U-duality superalgebra of [3]. Later on we shall take this for granted when trying to find the algebras corresponding to other string models at least for some degree to be determined. The divisor $D_{M2}$ of degree 3 corresponds to the $M2$ brane and $D_{M5}$ of degree 6 to the $M5$ brane. $D_{M2}$ and $D_{M5}$ verify the Serre duality equation $D_{M2} + D_{M5} = -K$ which corresponds to the electric-magnetic Hodge duality ($3 + 6 = d - 2 = 9$). We note that these divisors are effective and have vanishing genus. The correspondence between rational divisors of degree zero and BPS instantons (or rather the scalar fields to which they couple) was known for a long time [12, 13].

The duality of [3] is itself demystified if we define a Borcherds superalgebra by the simple fermionic superroot given by $\alpha_0 = H$. The Dynkin diagram, corresponding to the Cartan matrix $A = a = (\,-1\,)$ is given by:

◉ = **f**

$\alpha_0$

We give here our conventions for Dynkin diagrams of Borcherds superalgebras, the last character is a chemical label that will eventually dispense us from writing diagrams. The number of bonds between simple roots is the opposite of the off diagonal element of the (symmetrized) Cartan matrix, it is in effect an intersection number and the roots come in (at least) four colors, the dichotomy between real and imaginary is the naive one namely between strictly positive and negative $a_{ii}$'s:

| | | |
|---|---|---|
| ○ | Bosonic real root of length 2 | **B** |
| ⊗ | Bosonic imaginary root of length $\leq 0$ | **b** |
| ● | Fermionic "imaginary" root of length 1, $a_{ii} = 0$ | **F** |
| ◉ | Fermionic imaginary root of length $\leq -1$ | **f** |

**Table 1:** Labels for simple roots.

The positive superroots are fermionic and bosonic $\pi_n = nH$ with $n = 1$ or $n = 2$. By truncating to the set of positive superroots we obtain the following superalgebra

$$\{e_{\alpha_0}, e_{\alpha_0}\} = -e_{\pi_2}\,, \qquad [e_{\alpha_0}, e_{\pi_2}] = 0\,, \qquad [e_{\pi_2}, e_{\pi_2}] = 0\,. \tag{4.1}$$

The truncation to the finite dimensional superalgebra of [3] is defined by a $\mathbb{Z}$-gradation and is thus consistent, one may see it as a two step process namely restriction to positive degrees followed by truncation to some maximal degree at our disposal.

We have related three symmetries and the intersection form: the U-duality was known to be related to the part of the second cohomology orthogonal to the canonical class, the rest of the cohomology contains the extended superduality of [3] and corresponds actually to a Borcherds superalgebra of which the former is a truncation. Using chemical nomenclature for Dynkin molecules we may call the M theory Borcherds algebra: **f**. Actually in this case the superalgebra is finite dimensional and equal to OSp(1|2)!

In [3] the authors introduced a pseudo-involution $\mathcal{S}$ that exchanges the generators $e_{\pi_1}$ and $e_{\pi_2}$:

$$\mathcal{S}\, e_{\pi_N} = \pm e_{-\pi_N - K}\,, \qquad \mathcal{S}^2\, e_{\pi_N} = \epsilon_N e_{\pi_N}\,. \tag{4.2}$$

(There is a global sign arbitrariness which has no meaning but the relative $\epsilon_N$ is there to compensate for the square of Hodge dualisation that maybe equal to $-1$ for some values

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of the rank of the corresponding field strength $1 + \mathrm{grad}(\pi_N)$.) The operator $\mathcal{S}$ is well-defined even after truncation provided the spectrum of truncated positive superroots is invariant under Serre duality, this requires the appropriate choice of maximal degree: the spacetime dimension minus two. Note that here we have $\mathcal{S}^2\, e_{\alpha_i} = -e_{\alpha_i}$, so $\mathcal{S}^2 = -\mathrm{id}$. In lower dimensions we shall see that $\mathcal{S}^2$ acts sometimes as an involution, and sometimes as a pseudo-involution. Let us note that $\mathcal{S}$ does not preserve the commutation relations (4.1), up to signs it implements Serre duality.

Let us introduce the following nonlinear "potential" differential form:

$$\mathcal{V} = e^{A_{(3)}\, e_{\alpha_0}}\, e^{\widetilde{A}_{(6)}\, e_{\pi_2}}\,. \tag{4.3}$$

The Grassmann *angle* $A_{(3)}$ (resp. $\widetilde{A}_{(6)}$) is a 3-form (resp. 6-form) coupled to the M2- (resp. M6-) brane and defined on an eleven dimensional manifold. This manifold should be describable on the algebraic side of the correspondence, presently we can read from there its dimension $d$, the Hodge duality. We may conjecture that the spacetime coordinates will appear as algebraic moduli. Note that the generators $e_{\alpha_i}$ are even or odd according to whether the degrees of the associated field strengths are odd or even. The odd generator corresponds to what was called the 12th fermionic dimension in [3].

By an elementary calculation one checks that the field strength $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$ following from (4.3) is given by

$$\begin{aligned}\mathcal{G} &= dA_{(3)}\, e_{\alpha_0} + \left(d\widetilde{A}_{(6)} - \frac{1}{2}A_{(3)} \wedge dA_{(3)}\right) e_{\pi_2}\,,\\ &= F_{(4)}\, e_{\alpha_0} + \widetilde{F}_{(7)}\, e_{\pi_2}\,.\end{aligned} \tag{4.4}$$

Note that when the exterior derivative passes over a generator, the latter acquires a minus sign if it is odd. Thus $d(e_{\pi_i} A_n) = (-)^{\mathrm{grad}(\pi_i)} e_{\pi_i} dA_n$. We recall the (twisted) self-duality equation:

$$*\mathcal{G} = \mathcal{S}\,\mathcal{G}\,, \tag{4.5}$$

in components:

$$*F_{(4)} = \widetilde{F}_{(7)} \equiv d\widetilde{A}_{(6)} - \frac{1}{2}A_{(3)} \wedge F_{(4)}\,, \tag{4.6}$$

Since the doubled field strength $\mathcal{G}$ is written as $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$, it follows by taking an exterior derivative that we have the Cartan-Maurer equation $d\mathcal{G} = -d\mathcal{V} \wedge d\mathcal{V}^{-1} = d\mathcal{V}\,\mathcal{V}^{-1} \wedge d\mathcal{V}\,\mathcal{V}^{-1}$, and hence

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0\,. \tag{4.7}$$

Now, substituting (4.6) into (4.7), it follows that

$$d{*F_{(4)}} + \frac{1}{2}F_{(4)} \wedge F_{(4)} = 0\,. \tag{4.8}$$

This equation can be obtained by varying with respect to $A_{(3)}$ the bosonic lagrangian of eleven-dimensional supergravity [14] given by

$$\mathcal{L}_{11} = R\,{*\mathbb{1}} - \frac{1}{2}{*F_{(4)}} \wedge F_{(4)} - \frac{1}{6}F_{(4)} \wedge F_{(4)} \wedge A_{(3)}\,, \tag{4.9}$$

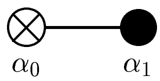
In [15] it has been shown that the superalgebra of gauge symmetries (4.1) implies nonlinear relations of the type $\frac{t_{\pi_1}}{2\pi} \cdot \frac{t_{\pi_1}}{2\pi} = \frac{t_{\pi_2}}{2\pi}$ where $t_{\pi_1}$ (resp. $t_{\pi_2}$) is the tension of the M2 brane (resp. M5 brane). More generally, for each non-vanishing commutator $[e_{\pi_i}, e_{\pi_j}] \sim e_{\pi_i+\pi_j}$ coming from a truncated Borcherds superalgebra, we can deduce a product relation between the tensions $\frac{t_{\pi_i}}{2\pi} \cdot \frac{t_{\pi_j}}{2\pi} = \frac{t_{\pi_i+\pi_j}}{2\pi}$. This correspondence has been recovered, on the del Pezzo side, in [1]. Applying the above rule gives also the Dirac-Nepomechie-Teitelboim quantisation condition. For BPS states tensions and charges are proportional to each other. These results have been reviewed in several Conference Proceedings to which we may refer [16]. As the truncated spectrum of positive roots is invariant by construction under the Serre duality, for each $\pi_i$, $-K - \pi_i$ will be a positive superroot. Another way to phrase this is to give a formula for $-K$ as a positive combination of simple roots. Here $-K = 3\alpha_0$.

### 4.2 IIA theory

Blowing up $\mathbb{P}^2$ in one point, we get the del Pezzo surface $\mathbb{B}_1$. If $E_{11}$ is the exceptional divisor corresponding to this blow up, the Picard lattice is now of rank two : $\mathrm{Pic}(\mathbb{B}_1) = \mathbb{Z}H + \mathbb{Z}E_{11}$. In this orthogonal basis the (unimodular) intersection matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From $\mathbb{B}_1$, we may retrieve in the same way as above type-IIA supergravity and its duality superalgebra as a truncated Borcherds algebra. The anticanonical divisor is $-K = 3H - E_{11}$. Divisors generating $\mathrm{Pic}(\mathbb{B}_1)$ are $\alpha_0 = H - E_{11}$ and $\alpha_1 = E_{11}$. Both have null virtual genus and minimal positive degrees $-K.\alpha_i$. In fact $-K = 3\alpha_0 + 2\alpha_1$.

They give the intersection matrix $A = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ and the Cartan matrix $a = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which satisfies Borcherds superalgebra axioms, with a Dynkin diagram given by:




and its formula is **bF**.

Chevalley-Serre-Kac-Moody-Borcherds relations allow us to construct the positive roots of this algebra, and by truncating to positive degrees lower than 9, we get table 2.

| Degree | positive root | BPS state | potential |
|---|---|---|---|
| 0 | 0 | ? | $\phi$ |
| 1 | $\alpha_1$ | D0 | $\mathcal{A}_{(1)}$ |
| 2 | $\alpha_0$ | F1 | $\mathcal{A}_{(2)}$ |
| 3 | $\alpha_0 + \alpha_1$ | D2 | $\mathcal{A}_{(3)}$ |
| 5 | $2\alpha_0 + \alpha_1$ | D4 | $\widetilde{A}_{(5)}$ |
| 6 | $2\alpha_0 + 2\alpha_1$ | NS5 | $\widetilde{A}_{(6)}$ |
| 7 | $3\alpha_0 + \alpha_1$ | D6 | $\widetilde{A}_{(7)}$ |
| 8 | $-K = 3\alpha_0 + 2\alpha_1$ | ? | $\psi$ |
| 9 | $4\alpha_0 + \alpha_1$ | D8 | none |

**Table 2:** IIA positive roots.

We remark that as far as the superalgebra structure is concerned we should actually choose maximal degree 8 and preserve symmetry under Serre duality $C \to -K - C$, this allows us to define the pseudo-involution (4.2) on the truncated positive superroot set. It corresponds on the associated forms to Hodge duality

$$d\mathcal{A}_{(i)} = \pm * d\widetilde{A}_{(8-i)} + \text{nonlinear terms}\,. \tag{4.10}$$

As there is no dynamical field coupled to the D8 brane it stays alone, but note that it is rational and secondly that it has no Serre dual in the root system. In other words rationality as we have seen is a Serre duality invariant concept whereas the existence of BPS states as we define them today is not.

We also note that most of these positive roots are rational curves (including the root corresponding to the D8 brane) and correspond to $\frac{1}{2}$-BPS states whose brane spacetime dimension is given by the degree, the only non rational curve is simply the Serre dual of zero, in other words the anticanonical divisor. Adding to the algebra generators $e_{\alpha_0}$ and $e_{\alpha_1}$ the Cartan element $h = \frac{1}{2}h_{\alpha_0} - h_{\alpha_1}$ and dropping the other Cartan generator as in M theory we get the superalgebra relations:

$$\begin{aligned} [h, e_{\alpha_1}] &= -\frac{3}{2}e_{\alpha_1}\,, \qquad [h, e_{\alpha_0}] = e_{\alpha_0}\,, \qquad [h, e_{\alpha_1+\alpha_0}] = -\frac{1}{2}e_{\alpha_1+\alpha_0}\,, \\ [h, e_{3\alpha_0+\alpha_1}] &= \frac{3}{2}e_{3\alpha_0+\alpha_1}\,, \quad [h, e_{2\alpha_0+2\alpha_1}] = -e_{2\alpha_0+2\alpha_1}\,, \quad [h, e_{2\alpha_0+\alpha_1}] = \frac{1}{2}e_{2\alpha_0+\alpha_1}\,. \end{aligned} \tag{4.11}$$

$$\begin{aligned} [e_{\alpha_1}, e_{\alpha_0}] &= -e_{\alpha_1+\alpha_0}\,, \qquad \{e_{\alpha_1}, e_{\alpha_1+2\alpha_0}\} = -e_{2\alpha_1+2\alpha_0}\,, \quad [e_{\alpha_0}, e_{\alpha_1+\alpha_0}] = -e_{\alpha_1+2\alpha_0}\,, \\ [e_{\alpha_0}, e_{\alpha_1+2\alpha_0}] &= -e_{\alpha_1+3\alpha_0}\,, \quad \{e_{\alpha_1+\alpha_0}, e_{\alpha_1+\alpha_0}\} = -e_{2\alpha_1+2\alpha_0}\,. \end{aligned} \tag{4.12}$$

$$\begin{aligned} \{e_{\alpha_1}, e_{\alpha_1+3\alpha_0}\} &= \frac{3}{8}e_{2\alpha_1+3\alpha_0}\,, \qquad [e_{\alpha_0}, e_{2\alpha_1+2\alpha_0}] = \frac{2}{8}e_{2\alpha_1+3\alpha_0}\,, \\ \{e_{\alpha_1+\alpha_0}, e_{\alpha_1+2\alpha_0}\} &= \frac{1}{8}e_{2\alpha_1+3\alpha_0}\,. \end{aligned} \tag{4.13}$$

As above, one can associate to this superalgebra the potential

$$\mathcal{V} = e^{\frac{1}{2}\phi h}\, e^{\mathcal{A}_{(1)}e_{\alpha_1}}\, e^{\mathcal{A}_{(2)}e_{\alpha_0}}\, e^{\mathcal{A}_{(3)}e_{\alpha_1+\alpha_0}}\, e^{\widetilde{A}_{(5)}e_{\alpha_1+2\alpha_0}}\, e^{\widetilde{A}_{(6)}e_{2\alpha_1+2\alpha_0}}\, e^{\widetilde{\mathcal{A}}_{(7)}e_{\alpha_1+3\alpha_0}}\, e^{\frac{1}{2}\psi e_{2\alpha_1+3\alpha_0}}\,. \tag{4.14}$$

and derive its field strength $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$. Let us comment that as in the previous case and in the sequel we do not see in the field theory description (even after doubling) any field for one of the Cartan generators, in the present case we use only the combination $h = \frac{1}{2}h_{\alpha_0} - h_{\alpha_1}$ for instance (we are dropping the $(-K)$-degrees above 8 and hope to return to this fact in a later work), nevertheless it corresponds to a trivial scaling symmetry of the field equations.

The Cartan-Maurer equation (4.7), combined with the self-duality condition (4.5), gives precisely the equations of motion for the bosonic part of the IIA 10-dimensional supergravity lagrangian

$$\begin{aligned} \mathcal{L}_{10} = {}& R{*}\mathbb{1} - \frac{1}{2}{*}d\phi \wedge d\phi - \frac{1}{2}He^{-\frac{3}{2}\phi} *\mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{2}e^{\phi} *F_{(3)} \wedge F_{(3)} - \\ & - \frac{1}{2}e^{-\frac{1}{2}\phi} *F_{(4)} \wedge F_{(4)} - \frac{1}{2}dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)}\,, \end{aligned} \tag{4.15}$$

where $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$, $F_{(3)} = dA_{(2)}$ and $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$.

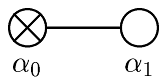
We have checked the roots corresponding to the fields of IIA SUGRA by using the denominator formula (see for instance [10]). We shall complete the same checks for IIB theory and M theory on $T^2$ below.

### 4.3 IIB theory

We are now ready for the most striking evidence for a Borcherds algebra in string theory. The generators are all bosonic (only even differential forms appear). The unique other smooth del Pezzo surface with D=10 is $\mathbb{P}^1 \times \mathbb{P}^1$ whose Picard group is generated by the two classes of $\mathbb{P}^1$'s $l_1$ and $l_2$. If we see $\mathbb{P}^1 \times \mathbb{P}^1$ as a trivial bundle with fiber $l_1$ and section $l_2$, we deduce that $l_1^2 = 0$ as two fibers do not intersect (similarly $l_2^2 = 0$) and $l_1.l_2 = 1$ as the section and one fiber intersect in one point. Following the line of what we have just done for $\mathbb{B}_1$, this surface will now correspond to type-IIB 10-dimensional supergravity. Divisors generating Pic($\mathbb{F}_0$) are $\alpha_0 = l_1$ and $\alpha_1 = l_2 - l_1$. Here $-K = 4\alpha_0 + 2\alpha_1$.

The Cartan matrix $a_{ij} = A_{ij}$ is $\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$.

Its associated Borcherds algebra has one lightlike simple root, it plays a crucial role in the construction of the Monster algebra [17] and has been analyzed in [2]. The (purely bosonic) Dynkin diagram is represented by:




with formula **bB**. Let us recall that the node $\alpha_1$ corresponds to the S-duality symmetry of IIB theory, which correponds to the permutation of the $\mathbb{P}^1$'s, it is different from the IIA node with the same label, in fact they will span two of the three dimensions of the reduction to 9 dimensions. $\alpha_1$ spans the part of the cohomology that is orthogonal to the canonical class $K$. Let us note also that we have broken the manifest (Weyl) symmetry between the two $\mathbb{P}^1$'s but only in the choice of simple roots.

The positive roots, of degree lower than 10, are in table 3.

It should be noted that the divisor $-\alpha_1$ which is a negative root has vanishing virtual genus and degree zero. It corresponds to the NS-1 instanton which is S-dual to D-1 and

| Degree | positive root | BPS state | $d$-form |
|---|---|---|---|
| 0 | 0 | ? | $\phi$ |
| 0 | $\alpha_1$ | D-1 | $\chi$ |
| 2 | $\alpha_0$ | F1 | $A^2_{(2)}$ |
| 2 | $\alpha_0 + \alpha_1$ | D1 | $A^1_{(2)}$ |
| 4 | $2\alpha_0 + \alpha_1$ | D3 | $B_{(4)}$ |
| 6 | $3\alpha_0 + \alpha_1$ | D5 | $\widetilde{A}^1_{(6)}$ |
| 6 | $3\alpha_0 + 2\alpha_1$ | NS5 | $\widetilde{A}^2_{(6)}$ |
| 8 | $4\alpha_0 + \alpha_1$ | D7 | $\widetilde{\chi}$ |
| 8 | $-K = 4\alpha_0 + 2\alpha_1$ | ? | $\psi$ |
| 8 | $4\alpha_0 + 3\alpha_1$ | NS7 | none |

**Table 3:** IIB positive roots.

completes the triplet of sources for the 3 fields of the (SO(2)) gauge invariant formulation of the scalar sector. We can now introduce the formal form $\mathcal{V}$ given by

$$\mathcal{V} = e^{\frac{1}{2}\phi\, h}\, e^{\chi\, e_{\alpha_1}}\, e^{(A^1_{(2)}\, e_{(\alpha_0+\alpha_1)} + A^2_{(2)}\, e_{\alpha_0})}\, e^{B_{(4)}\, e_{(2\alpha_0+\alpha_1)}} \times \\ \times e^{(\widetilde{A}^1_{(6)}\, e_{(3\alpha_0+\alpha_1)} + \widetilde{A}^2_{(6)}\, e_{(3\alpha_0+2\alpha_1)})}\, e^{\widetilde{\chi}\, e_{(4\alpha_0+2\alpha_1)}}\, e^{\frac{1}{2}\psi\, e_{(4\alpha_0+3\alpha_1)}}\,. \tag{4.16}$$

where $h = h_{\alpha_1}$.

One can define again an involution (4.2) such that the $2^{nd}$ order equations derived from (4.7) and (4.5) come from the bosonic part of the IIB 10-dimensional supergravity lagrangian. The positive root $-K$ has multiplicity one not two, it is Serre dual to the zero "root" with generator $h$.

### 4.4 M theory on $T^k, k \leq 8$

We consider $\mathbb{B}_k$ obtained by blowing up $k$ points in generic positions on $\mathbb{P}^2$. First, we will define a Borcherds superalgebra associated to the Picard lattice. The truncation to positive superroots whose degree is lower than $(9-k)$ will generate a superalgebra invariant under Serre duality as a linear space. This will permit us to define a pseudo-involution (4.2). The Maurer-Cartan equation (4.7) combined with the self-duality condition (4.5) will reproduce the equations of motion of M-theory compactified on a $k$-dimensional torus $T^k$. We note that the 3-dimensional Picard lattices of $\mathbb{B}_2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ blown up on a point are lorentzian, self-dual and odd. Consequently they are isomorphic, define the same Borcherds superalgebra and map to the same supergravity theory. This corresponds to the T-duality between IIA and IIB compactified on a circle.

The Picard lattice of $\mathbb{B}_k$ is generated by H (the class of a line in $\mathbb{P}^2$) and the exceptional divisors $E_{(11-i)}$, $0 \leq i \leq (k-1)$. We choose an appropriate basis $(\alpha_i,\, 0 \leq i \leq (k-2), \beta, \gamma)$ for this lattice which corresponds to the simple superroots of a Borcherds superalgebra. In table 4, we list the simple superroots with the associated Dynkin diagrams for all $k$'s.

The roots $(\alpha_i, \gamma)$ define a $\mathrm{sl}(k|1)$ superalgebra and the roots $(\alpha_i, \beta)$ represented by instantons, i.e. divisors with vanishing degree and virtual genus, define the Dynkin diagram of the U-duality group $E_k$.

The positive roots, of degree lower than $9-k$ $(3 \leq k \leq 8)$ whenever they exist are in table 5.

We have included the dilatons $\vec{\phi}$. These truncations define precisely the superalgebras which appeared in [3]. Defining the involution (4.2), the $2^{nd}$ order equations derived from (4.7) and (4.5) come from the bosonic part of the 11-dimensional supergravity lagrangian compactified on the $k$-dimensional torus $T^k$. We shall return to the denominator formula in a subsequent paper, for the time being we have mostly done random checks of the root structure beyond $k = 3$.

By construction all the intersection matrices are unimodular. Let us notice that the case of spacetime dimension 3 and Picard rank 9 has the odd simple root at the location of the affine root of $E_9$. Indeed $-K$ can be expressed as the sum of the most positive root of $E_8$ with this odd simple root, it is orthogonal to all roots of $E_8$. In fact this is probably true in all cases for $D = 3$. (We have checked it for all cases where the U-duality algebra is a simple Lie algebra.)



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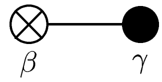
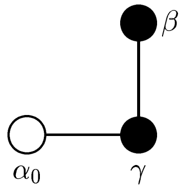
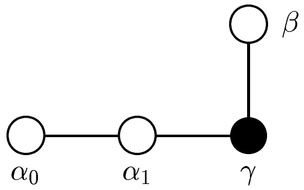
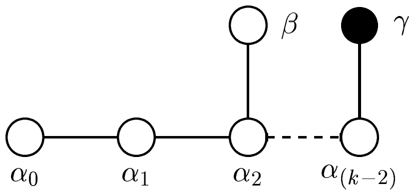
| $k$ | Simple superroot | Dynkin diagram | Formula |
|---|---|---|---|
| 0 | $\beta = H$ |  | **f** |
| 1 | $\beta = H - E_{11}$<br>$\gamma = E_{11}$ |  | **bF** |
| 2 | $\alpha_0 = E_{11} - E_{10}$<br>$\beta = H - E_{11} - E_{10}$<br>$\gamma = E_{10}$ |  | **BFF** |
| 3 | $\alpha_0 = E_{11} - E_{10}$<br>$\alpha_1 = E_{10} - E_9$<br>$\beta = H - E_{11} - E_{10} - E_9$<br>$\gamma = E_9$ |  | **BBFB** |
| 4 to 8 | $\alpha_i = E_{11-i} - E_{10-i}$<br>$0 \leq i \leq (k-2)$<br>$\beta = H - E_{11} - E_{10} - E_9$<br>$\gamma = E_{12-k}$ |  | **BB(BB)B·BF** |

**Table 4:** M-theory on $T^k$: Borcherds diagrams.

Now, one may consider $\mathbb{B}_9$, it is not a del Pezzo surface as $K^2_{\mathbb{B}_9} = 0$ instead $K_{\mathbb{B}_9}$ is "nef" (i.e. $K_{\mathbb{B}_9}.e \geq 0$ for any effective divisor on the surface). In an appropriate basis [18], the Picard lattice $\mathrm{Pic}(\mathbb{B}_9)$ has intersection form

$$-\Gamma_{E_8} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\Gamma_{E_8}$ the Cartan matrix of the exceptional Lie algebra $E_8$. The formula for its diagram is $\mathbf{B_2(BB)B_5F}$, it comes from a unimodular lattice of (maximal) rank 10 to be compared to the even unimodular lattice $E_{10}$: $\mathbf{B_2(BB)B_6}$.

## 5. Type-I/heterotic

We know that 10-dimensional type-IIA and-IIB theories correspond to smooth del Pezzo surfaces $\mathbb{B}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Is there any surface whose associated Borcherds algebra gives type-I or heterotic theories? Our answer is that it is possible to retrieve the truncated version of these theories where the gauge sector has been dropped in such a way that their massless sector is type-I supergravity.

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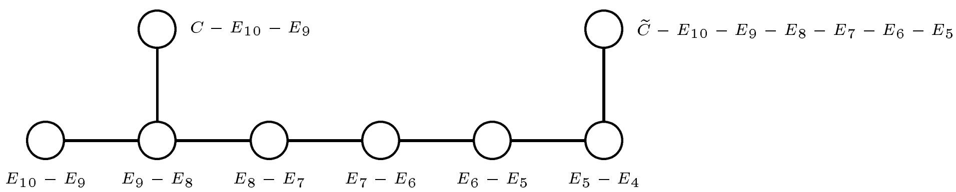
| Degree | positive root | BPS state | $d$-form |
|---|---|---|---|
| 0 | 0 | dilatons | $\vec{\phi}$ |
| 0 | $H - E_i - E_j - E_l$ | thrice-wrapped M2 | $A_{(0)ijl}$ |
| 0 | $E_i - E_j i > j$ | Kaluza-Klein modes | $\mathcal{A}^i_{(0)j}$ |
| 1 | $E_i$ | momentum | $\hat{\mathcal{A}}^i_{(1)}$ |
| 1 | $H - E_i - E_j$ | twice-wrapped M2 | $A_{(1)ij}$ |
| 2 | $H - E_i$ | once-wrapped M2 | $A_{(2)i}$ |
| 3 | $H$ | M2 | $A_{(3)}$ |
| $6-k$ | $2H - \sum_p E_p$ | $(k)$-wrapped M5 | $\widetilde{A}_{(6-k)}$ |
| $7-k$ | $2H - \sum_{p \neq i} E_p$ | $(k-1)$-wrapped M5 | $\widetilde{A}^i_{(7-k)}$ |
| $8-k$ | $2H - \sum_{p \neq i\, j} E_p$ | $(k-2)$-wrapped M5 | $\widetilde{A}^{ij}_{(8-k)}$ |
| $8-k$ | $3H - \sum_{p \neq i} E_p - 2E_i$ | magnetic dual of momentum | $\widetilde{\mathcal{A}}_{(8-k)i}$ |
| $9-k$ | $3H - \sum_{p \neq i\, j} E_p - 2E_i$ | magnetic dual of Kaluza-Klein | $\widetilde{\mathcal{A}}^j_{(9-k)i}$ |
| $9-k$ | $2H - \sum_{p \neq i\, j\, l} E_p$ | $(k-3)$-wrapped M5 | $\widetilde{A}^{ijl}_{(9-k)}$ |
| $9-k$ | $-K = 3H - \sum_p E_p$ | magnetic dual of dilatons | $\vec{\psi}$ |

**Table 5:** M-theory on $T^k$: positive roots.

### 5.1 Physical requirements

It is well known that type-I and heterotic theories, deprived of their gauge sector and compactified on tori $T^k$, $k \geq 3$, have U-duality Lie groups $A_1 \times A_1$, $A_3$, $D_4$, $D_5$, $D_6 \times A_1$, $D_8$ instead of the $E_k$ sequence of type II.

The 10-dimensional type-I theory has as BPS states the D1 and D5 branes, while heterotic theories have the fundamental string F1 and the NS5 brane. When compactifying on tori, BPS states corresponding to those objects may be wrapped on tori. Instantons correspond to the simple roots of the U-duality algebra, and one sees that the $A_1$ root which appears in 8 dimensions comes from a twice wrapped 2-brane whereas the $A_1$ root appearing in 4 dimensions comes from a fully wrapped 5-brane. This is shown in the following Lie Dynkin diagram, where simple roots involving $E_i$ progressively appear in dimensions lower than $i$.




We deal here with the type-I case, but the story is the same for the heterotic case, if one exchanges D1 and D5 for F1 and NS5. Writing $C$ and $\widetilde{C}$ for the rational divisors corresponding to D1 and D5 branes, the BPS states and their corresponding rational divisors for the $k^{th}$ toroidal compactification are presented in table 6, where the $E_i$'s are the exceptional $(-1)$-curves associated with the successive compactifications.

| Degree | positive root | BPS state | $d$-form |
|---|---|---|---|
| 0 | 0 | dilatons | $\vec{\phi}$ |
| 0 | $C - E_i - E_j$ | twice-wrapped D1 | $A_{(0)ij}$ |
| 0 | $E_i - E_j$ | Kaluza-Klein modes | $\mathcal{A}^i_{(0)j}$ |
| 1 | $E_i$ | momentum | $\hat{\mathcal{A}}^i_{(1)}$ |
| 1 | $C - E_i$ | once-wrapped D1 | $A_{(1)i}$ |
| 2 | $C$ | D1 | $A_{(2)}$ |
| $6-k$ | $\widetilde{C} - \sum_p E_p$ | $(k)$-wrapped D5 | $\widetilde{A}_{(6-k)}$ |
| $7-k$ | $\widetilde{C} - \sum_{p\neq i} E_p$ | $(k-1)$-wrapped D5 | $\widetilde{A}^i_{(7-k)}$ |
| $7-k$ | $-K - E_i$ | magnetic dual of momentum | $\widetilde{\mathcal{A}}_{(7-k)i}$ |
| $8-k$ | $-K - E_i + E_j$ | magnetic dual of Kaluza-Klein | $\widetilde{\mathcal{A}}^j_{(8-k)i}$ |
| $8-k$ | $\widetilde{C} - \sum_{p\neq i\,j} E_p$ | $(k-2)$-wrapped D5 | $\widetilde{A}^{ij}_{(8-k)}$ |
| $8-k$ | $-K$ | magnetic dual of dilatons | $\vec{\psi}$ |

**Table 6:** Truncated I on $T^k$.

### 5.2 Ampleness of $-K$

We would like now to generalize the correspondence between algebraic surfaces and string theories from M-theory to the 16 supercharges case. Let us try at first to work with any ample divisor. It will be needed to define a degree and a projective embedding that guarantees the isomorphism between Cartier classes and the Picard group. The degree of a divisor is given by its scalar product with the ample divisor $H_k$. For a divisor corresponding to a BPS state we shall assume it is its physical spacetime dimension. In particular, we have $H_0.C = 2$. From the degrees of $E_i$ and $E_i - E_j$ we deduce that when we blow up the $k^{th}$ point, we have $H_k = H_{k-1} - E_{11-k}$. We also know from general properties of the blowing-up operation that $K_k = K_{k-1} + E_{11-k}$.

$-K_k$ must be Cartier of degree $8 - k = D - 2$ because we assume Serre duality which corresponds in the physical space-time to the Hodge duality between $p$-forms and $(D - 2 - p)$-forms, this is by definition the Gorenstein property. The D1 and D5 branes being Hodge duals in 10 dimension, we must take $\widetilde{C} = -K_0 - C$.

As the $E_i$'s are exceptional curves that appear when blowing up, they are orthogonal to each other and to $C$ and $\widetilde{C}$. $C - E_{10} - E_9$ is a Lie simple root, it must be of self-intersection $-2$ like $E_i - E_j$. It follows that $C^2 = 0$. From the Lie root $\widetilde{C} - E_{10} - E_9 - E_8 - E_7 - E_6 - E_5$ we get similarly $\widetilde{C}^2 = 4$. From the Adjunction formula applied to the rational divisors $C$ and $\widetilde{C}$ we now get $C.K_0 = -2$ and $\widetilde{C}.K_0 = -6$ by assuming these divisors are still rational.

In summary, we must have

$$H_0 \cdot C = 2 = -K_0 \cdot C \tag{5.1}$$

$$H_0 \cdot \widetilde{C} = 6 = -K_0 \cdot \widetilde{C} \tag{5.2}$$

The moduli space of 10-dimensional type-I theory has dimension 2, and so must the Picard lattice of our surface X. We see from $C^2 = 0$ and $\widetilde{C}^2 = 4$ that $C$ and $\widetilde{C}$ are linearly independent, and we can conclude that $H_0 = -K_0$ in Pic$(X)$, which gives $H_k = -K_k$ for compactifications. In particular the anticanonical classes $-K_k$ must be ample, which leads to the important conclusion that these surfaces must be del Pezzo.

### 5.3 Finding the surface

Du Val [19], Demazure [20] and Hidaka-Watanabe [21] classified all Picard rank 2, normal, Gorenstein del Pezzo's, which are in finite number. In particular for $D = K^2 + 2 = 10$ there are exactly three such surfaces, none of them having a Picard group corresponding to (truncated) type-I (or heterotic) theory. It is important to note that the algebraic surface we are looking for should not have any other rational curve of degree between 0 and 10 than those listed in the above table, which excludes $\mathbb{P}^1 \times \mathbb{P}^1$ which satisfies all the other requirements.

So if our surface exists, it must be a nonnormal (Gorenstein) del Pezzo. We recall that *nonnormal* means that there is a line singularity, and *Gorenstein* means that the anticanonical class $-K$ exists as a Cartier divisor.

One finds in Miles Reid's classification of Gorenstein nonnormal del Pezzo's [22] two surfaces satisfying all the desired properties. Let us describe the first one. (The second is a kind of degenerate case of it and has the same Picard group.)

The rational scroll $Y := \mathbb{F}_{a;b}$ [23] is defined as $\mathbb{C}^2 \times \mathbb{C}^2$ modded out by the two equivalence relations

$$(t_1, t_2, x_1, x_2) \sim (t_1, t_2, \mu x_1, \mu x_2) \tag{5.3}$$
$$(t_1, t_2, x_1, x_2) \sim (\lambda t_1, \lambda t_2, \lambda^{-a-b} x_1, \lambda^{-b} x_2)\,. \tag{5.4}$$

It is clear that the first relation makes $\mathbb{F}_{a;b}$ a $\mathbb{P}^1$-bundle. Its base is also isomorphic to a $\mathbb{P}^1$, the projection being defined by the ratio $t_1 : t_2$. It can be embedded in $\mathbb{P}^{a+2b+1}$ in the following way:

$$(t_1, t_2, x_1, x_2) \mapsto (t_1^{a+b}\, x_1, t_1^{a+b-1} t_2\, x_1, \ldots, t_2^{a+b}\, x_1, t_1^b\, x_2, t_1^{b-1} t_2\, x_2, \ldots, t_2^b\, x_2)\,. \tag{5.5}$$

The Picard group of the surface $Y$ is generated by two irreducible divisor classes: the fiber $A$, of self-intersection 0 and a $(-a)$-curve $B$, defined by $x_1 = 0$, which is a section of the bundle and therefore has intersection 1 with $A$.

Let us now take $a = 4$ and $b = 2$. $B$ defines a plane in $\mathbb{P}^9$ described by the vanishing of all coordinates but the last three. Then let us pick a point in this plane which is not on $B$: $(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$. Projecting $\mathbb{P}^9$ on $\mathbb{P}^8$ from this point, the image $X$ of $\mathbb{F}_{4;2}$ in $\mathbb{P}^8$ is given by

$$(t_1, t_2, x_1, x_2) \mapsto (t_1^6\, x_1, t_1^5 t_2\, x_1, \ldots, t_2^6\, x_1, t_1^2\, x_2, t_2^2\, x_2)\,. \tag{5.6}$$

Our surface $X$ is isomorphic to $\mathbb{F}_{4;2}$ except for the conic $B$ which is mapped to a double line. Still denoting by $A$ and $B$ the images of these curves in $X$, the Cartier divisors of X are generated by $B$ and $2A$. A divisor is Cartier if it can locally be described as the intersection of the surface with an hyperplane in $\mathbb{P}^n$, and one sees that the intersection of the surface with an hyperplane transverse to the double locus contains two fibers $A$, so that $A$ is not Cartier but $2A$ is.

The intersection of Cartier divisors of X is defined via its normalisation the normal surface $\mathbb{F}_{4;2}$. The anticanonical class is $-K_X = 6A + B$ and is ample. Rational divisors (defined as divisors of vanishing virtual genus) of nonnegative degree lower than 10 are $C = 2A$ and $\widetilde{C} = B + 4A$ which correspond respectively to the D1 and D5 branes, and one can easily verify that they satisfy all required properties. In particular, blowing up points in generic positions gives the surfaces corresponding to toroidal compactifications of this truncated type-I theory.

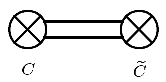
### 5.4 Borcherds algebras

We get from the surface just described the Borcherds algebras of these compactifications, which are described by the following Dynkin diagrams. Root lengths are easy to calculate, if one remembers that $C^2 = 0$, $\widetilde{C}^2 = 4$, $C \cdot \widetilde{C} = 2$, $E_i^2 = -1$, $E_i \cdot E_j = 0$ if $i \neq j$ and $E_i \cdot C = E_j \cdot \widetilde{C} = 0$. In dimension 10 the Cartan matrix restricted to CaCl(X) is the non unimodular but even and bosonic:

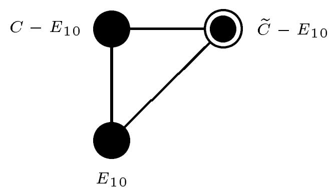
$$\begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix}.$$

We find the following Dynkin diagrams:

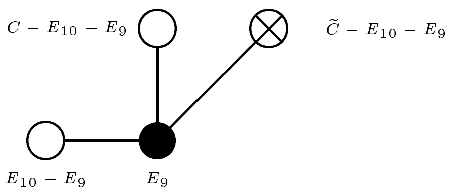
$D = 10$



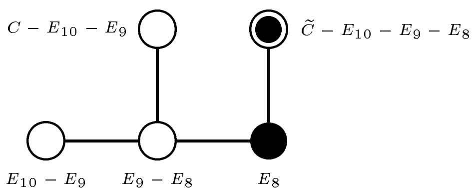

$D = 9$



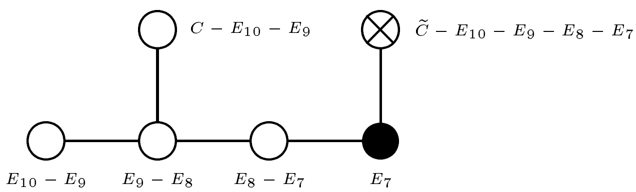

$D = 8$

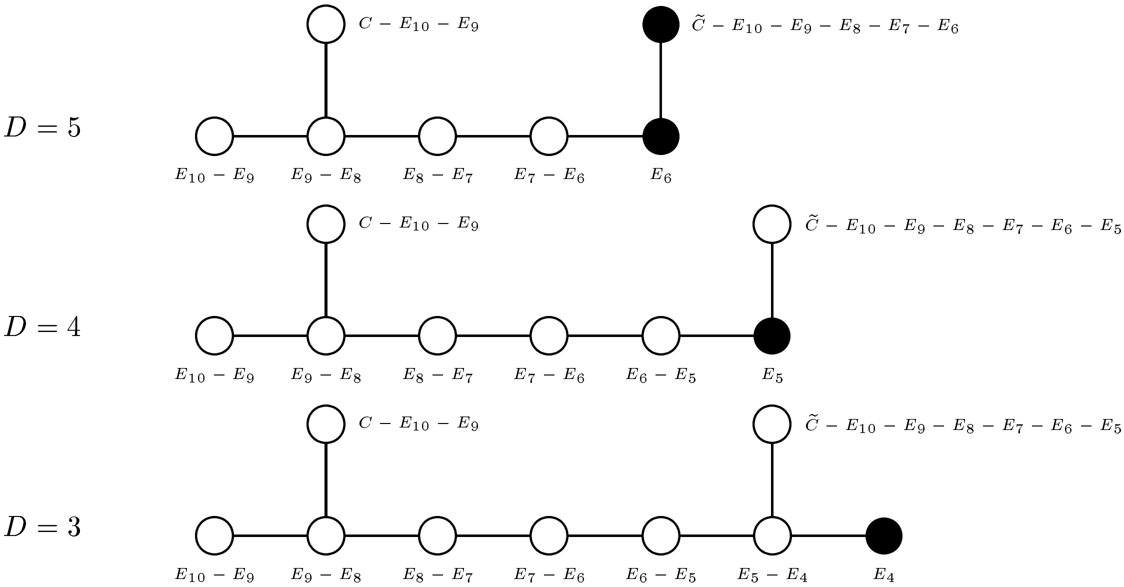



$D = 7$




$D = 6$






In this section the norms of the imaginary roots and superroots can be smaller than their values elsewhere in the paper, namely 0 and $-1$. Let us note also that in $D = 3$ the anticanonical class is again the sum of the odd simple root and the most positive root of the Lie algebra $D8$ of U-dualities.

As we said above, the story is the same for heterotic theories.

### 5.5 Fixed subalgebras of automorphisms of IIA/IIB Borcherds algebras

In 10 dimensions, the Borcherds algebra associated to (truncated) heterotic supergravity can be obtained even more easily as the fixed point subalgebra of the following automorphisms acting on the Borcherds superalgebra of IIA theory:

$$\begin{aligned} e_{\alpha_0} &\to +e_{\alpha_0} \\ e_{\alpha_1} &\to -e_{\alpha_1} \, . \end{aligned} \tag{5.7}$$

This involution preserves the potential $\mathcal{V}$ if $\mathcal{A}_{(1)} \to -\mathcal{A}_{(1)}$ and $\mathcal{A}_{(2)} \to \mathcal{A}_{(2)}$, which corresponds to the duality between M-theory compactified on $S^1/\mathbb{Z}_2$ and heterotic theory. A similar map can be found between IIB supergravity and (truncated) type-I by considering on the IIB Borcherds algebra the automorphism:

$$\begin{aligned} e_{\alpha_0} &\to +e_{\alpha_0} \\ e_{\alpha_1} &\to -e_{\alpha_1} \, . \end{aligned} \tag{5.8}$$

This corresponds to the orientifold projection. There is a S-dual version of that which maps to the truncated heterotic theory:

$$\begin{aligned} e_{\alpha_0} &\to -e_{\alpha_0} \\ e_{\alpha_1} &\to -e_{\alpha_1} \, . \end{aligned} \tag{5.9}$$

We should note that the twisted sectors do not appear and we obtain the type-I or heterotic theories without gauge sector. It is not clear, at present, how to encode the gauge group on the del Pezzo side.

| | $n=8$ | $n=7$ | $n=6$ | $n=5$ | $n=4$ | $n=3$ | $n=2$ | $n=1$ | $n=0$ |
|---|---|---|---|---|---|---|---|---|---|
| $D=11$ | $+$ | | | | | | | | |
| $D=10$ | $\mathbb{R}\, or\, A_1$ | $+$ | | | | | | | |
| $D=9$ | $\mathbb{R}\times A_1$ | $\mathbb{R}$ | | | | | | | |
| $D=8$ | $A_1\times A_2$ | $\mathbb{R}\times A_1$ | $A_1$ | | | | | | |
| $D=7$ | $E_4$ | $\mathbb{R}\times A_2$ | $\mathbb{R}\times A_1$ | $\mathbb{R}$ | $+$ | | | | |
| $D=6$ | $E_5$ | $A_1\times A_3$ | $\mathbb{R}\times A_1^2$ | $\mathbb{R}^2$ | $\mathbb{R}$ | | | | |
| $D=5$ | $E_6$ | $A_5$ | $A_2^2$ | $\mathbb{R}\times A_1^2$ | $\mathbb{R}\times A_1$ | $A_1$ | | | |
| $D=4$ | $E_7$ | $D_6$ | $A_5$ | $A_1\times A_3$ | $\mathbb{R}\times A_2$ | $\mathbb{R}\times A_1$ | $\mathbb{R}$ | $+$ | |
| $D=3$ | $E_8$ | $E_7$ | $E_6$ | $E_5$ | $E_4$ | $A_1\times A_2$ | $\mathbb{R}\times A_1$ | $\mathbb{R}\, or\, A_1$ | $+$ |

**Table 7:** Disintegration (i.e. oxidation) for $E_n$ cosets.

## 6. Magic triangle

### 6.1 The triangle

We have shown that the smooth surface $\mathbb{B}_k$ corresponds to M theory compactified on a $k$ torus. We have also seen that the dimensional reduction of eleven-dimensional supergravity to $D=3$ gives rise to a scalar coset theory with an $E_8$ global symmetry. In [24], the oxidation endpoints of three-dimensional symmetric space scalar theories $G/KG$ coupled to gravity have been determined in particular for all split subgroups $G$ of split $E_8$. Split is called maximally noncompact in some of the Physics literature, for instance $E_5 = \mathrm{SO}(5,5)$ or $E_4 = \mathrm{SL}(5,\mathbb{R})$.

By the *oxidation endpoint* of a three-dimensional scalar coset, we mean the bosonic theory in the highest possible dimension whose toroidal dimensional reduction gives back precisely the three-dimensional scalar model. It has been shown that the oxidation endpoint dimension for the subgroup $E_n$ with $2 \le n \le 8$ is usually given by $D_{max} = n+2$ or $n+3$. The oxidation sequence is presented in table 7 and a *magic* reflection symmetry across the diagonal appears.

Each vertical step down corresponds to compactification on a circle. In all cases, the oxidation endpoint theory includes the metric, a dilaton and a 3-form potential, and in $D \le 7$ there are no additional field potentials. In $D=10$, corresponding to $E_8$, there is also a 2-form potential and a vector potential. In $D=9$, corresponding to $E_7$, there is just an additional vector potential. In $D=8$, the $E_6$ column has an additional 0-form potential, or axion. There are, as we saw, three special cases that arise. For $E_8$ the "endpoint" implied by the generic discussion, namely $D=10$, can be further oxidized to the bosonic sector of $D=11$ supergravity. For $E_7$, the generic discussion leads to an endpoint in $D=9$, but again a further oxidation is possible, giving in this case a truncation of type-IIB supergravity in $D=10$ to the metric plus the self-dual 5-form. One could have predicted a bifurcation above the case $n=7$ and $D=6$, it is actually not there. A third special case is $E_4$, for which the "endpoint" in $D=6$ can be further oxidized to pure gravity in $D=7$, after first dualising the 3-form potential to a vector in $D=6$.

### 6.2 Normal Gorenstein del Pezzo surfaces

Normal Gorenstein del Pezzo surfaces have been studied by Du Val [19], Demazure [20] and the classification is given in a theorem of Hidaka and Watanabe [21]:

Denoting by $\pi : \tilde{X} \to X$ a minimal resolution of $X$, a normal del Pezzo surfaces such that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ (a natural assumption of connectedness) we have:

(i) $3 \leq D \leq 11$

(ii) X is smooth or the singular points of X are rational double points

(iii) if $D = 11$ then $X \simeq \mathbb{P}^2$

(iv) if $D = 10$ then either $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $X \simeq \mathbb{B}_1$ or $X$ is the cone over a quadric in $\mathbb{P}^2$

(v) if $3 \leq D \leq 9$, then there exists a set $\Sigma$ of points on $\mathbb{P}^2$ such that the points of $\Sigma$ are in almost general position, $|\Sigma| = 11 - D$ and $\tilde{X} = V(\Sigma)$. In this case, the resolution $\pi$ is the contraction of all curves on $\tilde{X}$ with self-intersection $-2$.

The singularities involved here are "Du Val singularities" i.e. they are singular points resulting from the contraction of a set of intersecting $(-2)$-curves. Such a singularity is characterized by a Dynkin diagram which describes the configuration of the contracted curves and belongs to the ADE classification [23]. Normal del Pezzo surfaces of Picard rank one or two are almost uniquely determined by their singularity type and are classified in [25, 26, 27].

We claim here that the supergravity theories of the magic triangle correspond to Gorenstein normal del Pezzo surfaces with one $A_p$ singular point. Precisely, the surfaces of the triangle are exactly the rank one and rank two Gorenstein normal del Pezzo surfaces with one $A_p$ singularity ($p = 8 - n$) and their generic blow up's.

Let us look first at Picard rank one surfaces with such a singularity. Five of them have $A_p$ singularity, with $p = 0, 1, 4, 7$ and 8.

For $p = 0$, there is no singularity: it is simply $\mathbb{P}^2$, with $K^2 = 9$, and we have already seen that it corresponds to eleven-dimensional supergravity. We have also seen that blowing up points in general position gives surfaces corresponding to toroidal compactifications of this theory.

The $p = 1$ case is the cone over a quadric in $\mathbb{P}^2$, which has indeed $A_1$ singularity and has $K^2 = 8$. It corresponds to the truncation of type-IIB supergravity in $D = 10$ to the metric plus the self-dual 5-form. We have $\tilde{X} \simeq \mathbb{F}_2$ and the resolution $\pi$ is given by contracting the minimal section of $\tilde{X}$. The Hirzebruch surface $\mathbb{F}_2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Its Picard lattice is generated by its minimal section $E$, such that $E^2 = -2$, and a fiber $C$. We know from its bundle structure that $C^2 = 0$ and $E.C = 1$. The anticanonical divisor can be found using the adjunction formula: $-K = 2E + 4C$ and one may verify that $K^2 = 8$. The Picard lattice of the cone after contraction of the $(-2)$-curve $E$ is the sublattice orthogonal to $E$ in $\mathrm{Pic}(\mathbb{F}_2)$. It is generated by the unique rational curve $E + 2C$, and the anticanonical divisor $-K$ is unchanged. This rational curve of degree 4 may correspond to a 3-brane $B3$ coupled to a self-dual 5-form on the field theory side. Now, as in the previous section, we can construct the rank one Borcherds algebra whose truncation should be a symmetry of the corresponding field theory. Then, the whole vertical column $n = 7$ can be obtained by blowing up generic points on this quadric cone, which still corresponds to toroidal compactifications of the $D = 10$ theory.

For $p = 4$, we start from $\mathbb{P}^2$ and blow up a point on it. Then take a point on the exceptional curve of this blow up and blow it up into a new exceptional curve. Now take again a point on this one, such that the three considered points sit on a line, and blow it up. After blowing up a fourth point taken on the last exceptional curve, we have a surface with four $(-2)$-curves in $A_4$ configuration, which is not del Pezzo. But contracting this bunch of $(-2)$-curves we get a singular del Pezzo surface of type $A_4$ with a Picard lattice of rank one. The Picard lattice is easy to compute: if as usual $H$ is a line in $\mathbb{P}^2$ and $E_{11}$, $E_{10}$, $E_9$ and $E_8$ are the four exceptional curves. Now we have also four $(-2)$-curves representing the classes of $E_{11} - E_{10}$, $E_{10} - E_9$, $E_9 - E_8$ and $H - E_{11} - E_{10} - E_9$ and one has $K = -3H + E_{11} + E_{10} + E_9 + E_8$ ($K^2 = 5$) as in the regular case. The Picard lattice of the final surface is the sublattice orthogonal to the four $(-2)$-curves and is generated by $-K$, which is not changed by this contraction (Du Val singularities have crepant resolutions in technical terms). $K^2 = 5$ tells us that this surface corresponds to a $D = 7$ theory, which is pure gravity.

The next two cases are constructed in the same way. For $p = 7$, one blows up seven points which are taken infinitely close as above, such that the six divisors $E_{11} - E_{10}$ to $E_6 - E_5$ are $(-2)$-curves; the first six points must also lie on a conic $2H$, so that $2H - E_{11} - E_{10} - E_9 - E_8 - E_7 - E_6$ is also a $(-2)$-curve. All these $(-2)$-curves form an $A_7$ diagram, and after contraction of them we have a del Pezzo surface with $A_7$ singularity and a Picard lattice generated by $-K = 3H - E_{11} - E_{10} - E_9 - E_8 - E_7 - E_6 - E_5$. As $K^2 = 2$, it corresponds to a $D = 4$ theory, which is simply four-dimensional gravity.

For $p = 8$, the story is the same with eight infinitely close points blown up such that $E_{11} - E_{10}$ to $E_5 - E_4$ are $(-2)$-curves and $3H - 2E_{11} - E_{10} - E_9 - E_8 - E_7 - E_6 - E_5 - E_4$ is too. This last condition means that there is conic $3H$ which passes through all the considered points and has the first one as a double point. Contracting all these $(-2)$-curves, one gets this time an $A_8$-singular del Pezzo surface with Picard lattice generated by $-K$, with $D = K^2 + 2 = 3$.

We may note the general formula $K^2 = n + 1$ in the minimal rank one case. Let us now look at Picard rank two del Pezzo surfaces with $A_p$ singularity. Some of them are simply obtained by blowing up a generic point on one of the Picard rank one surfaces just described. There is also $\mathbb{P}^1 \times \mathbb{P}^1$ which corresponds to IIB supergravity as we have seen. The other ones are obtained by blowing up $p+1$ infinitely close points on $\mathbb{P}^2$, each one lying on the exceptional curve of the preceding blow-up, so that $E_i - E_{i-1}$'s ($11 - p \leq i \leq 11$) form an $A_p$ sequence of $(-2)$-curves which are then contracted. One gets thus del Pezzo surfaces with $A_p$ singularity, Picard rank two and $D = K^2 + 2 = 10 - p = n + 2$.

All other surfaces of the magic triangle are obtained by blowing up those of Picard rank one and two, $A_p$-singular, del Pezzo surfaces, which are still related to toroidal compactifications of the corresponding field theories. One might expect that in a given column blowing up points on del Pezzo's of Picard rank one and two generates two sequences of surfaces. But this is not the case: we already know that IIA ($\mathbb{B}_1$) and IIB ($\mathbb{P}^1 \times \mathbb{P}^1$) give the same surface when compactified on tori. Similarly, blowing up a generic point on the quadric cone for $n = 7$ and on the $A_4$ Picard rank one del Pezzo gives respectively the same surfaces as the Picard rank two surfaces described in the previous paragraph with $p = 1$ and $p = 4$. Actually we have general relations:

$$D - 2 = K^2 = n + 1 - k - r = 9 - r - (k + p)\,,$$

where $r$ is the rank of the Picard group after blowing down all $-1$-curves.

There are three more cases with two different del Pezzo's beyond those corresponding to U-duality groups $A_1$ and $\mathbb{R}$: $n = 8$, $D = 10$ and $n = 1$, $D = 3$. One can indeed get $A_2$ instead of $\mathbb{R} \times A_1$ in $n = 7$, $D = 8$ and $n = 3$, $D = 4$, and one can have $A_1^2$ in addition to the $\mathbb{R}^2$ case in $n = 5$, $D = 6$.

### 6.3 Symmetric triangles

The symmetry of the triangle with respect to reflection across the diagonal relates two field theories with identical U-duality groups. On the geometric side, this is a symmetry between the set of $(-2)$-divisors of degree 0, and therefore between the Weyl groups they generate. Actually, we can prove that the sublattices orthogonal to $-K$ in the Picard lattices of any pair of symmetric surfaces of the triangle are identical.

This can be explained by the fact that on a del Pezzo surface of degree $K^2 = 1$, the orthogonal to $n$ mutually orthogonal $(-1)$ rational divisors and the orthogonal to an $A_n$ chain of $(-2)$ rational divisors are the same, when we restrict attention to divisors orthogonal to $-K$ in the Picard lattice. Therefore contracting $n$ orthogonal $(-1)$-curves and taking the orthogonal to an $A_k$ chain of $(-2)$-divisors gives the same degree 0 part of the Picard lattice as what one gets with $n$ and $k$ exchanged. Consequently the $K^\perp$'s are the same for the del Pezzo of degree $(1 + n)$ with an $A_k$ singularity and the del Pezzo of degree $(1 + k)$ with singularity $A_n$.[1]

This argument applies to any del Pezzo of degree one and not only to the smooth one, so one can construct a similar triangle starting from any of them, for instance there is a type-I magic triangle, we shall return to these in a subsequent publication.

### 6.4 Dynkin diagrams

We give here Borcherds superalgebras attached to the $A_p$-singular normal del Pezzo's from which (the bosonic part of) field theories can be retrieved ($p = 8 - n$). In order to calculate the lengths of simple roots, one must know that $B_i^2 = i - 1$ (these divisors correspond to $i$-branes). When the only root of the Dynkin diagram is $-K$, its multiplicity must be set to zero to be consistent with Hodge duality.

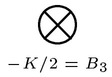
[1] A. Keurentjes informed us that he could trace the symmetry of the triangle to the possibility of extracting a linear subgroup of dualities from each side of the triangle, we independently proved the symmetry from algebraic geometry and related together three different mechanisms, the singular blow-downs, the regular blow-downs and the older observation of linear subgroups from dimensional reduction.



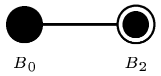
JHEP04(2002)049

### 6.4.1 $n = 7$

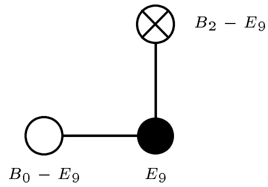
$D = 10$



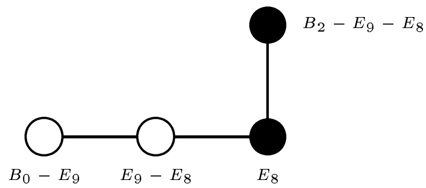

$D = 9$



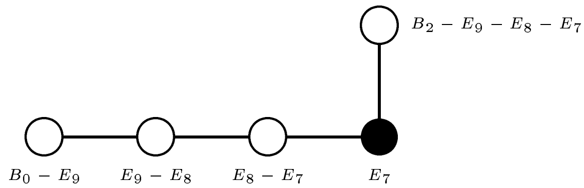

$D = 8$



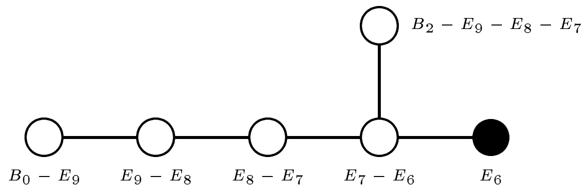

$D = 7$



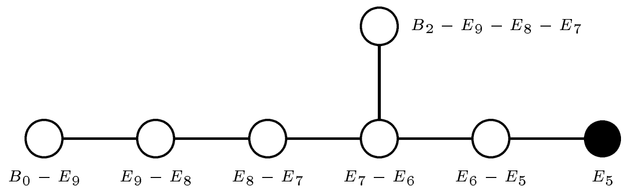

$D = 6$



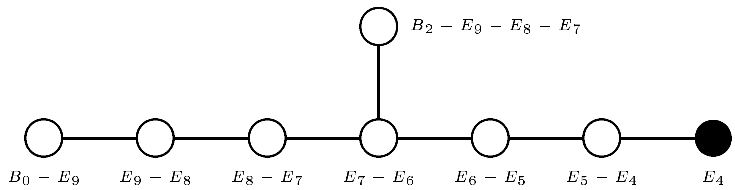

$D = 5$




$D = 4$

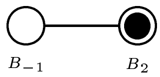



$D = 3$

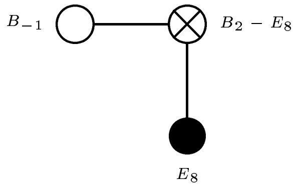


### 6.4.2 $n = 6$

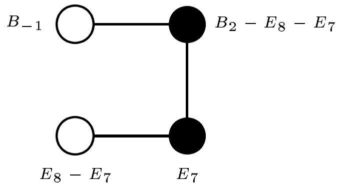
$D = 8$



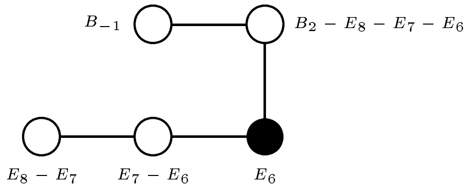

$D = 7$



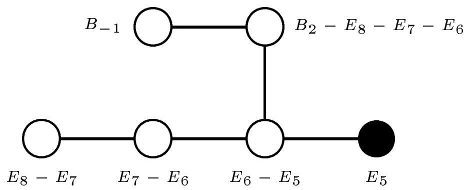

$D = 6$



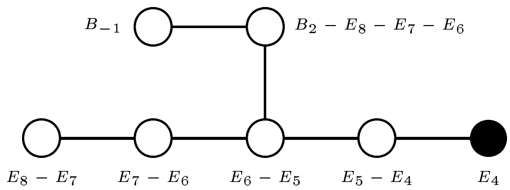

$D = 5$




$D = 4$

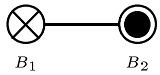



$D = 3$

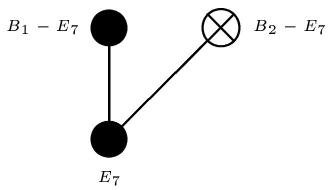



### 6.4.3 $n = 5$

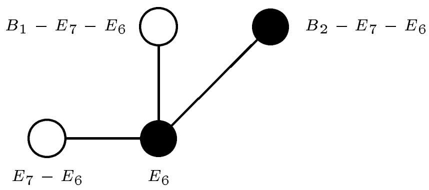
$D = 7$



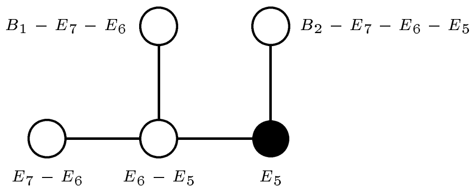

$D = 6$



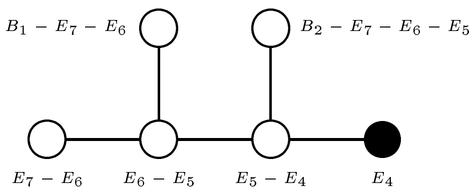
$D = 5$




$D = 4$



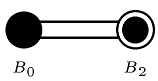

$D = 3$



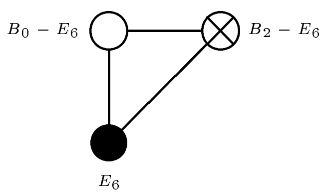

#### 6.4.4 $n = 4$

$D = 7$

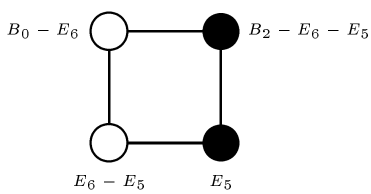

$D = 6$



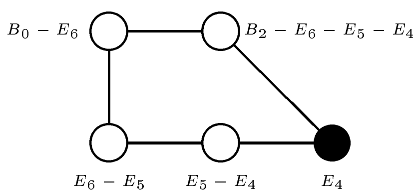

$D = 5$




$D = 4$

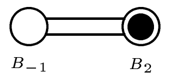



$D = 3$

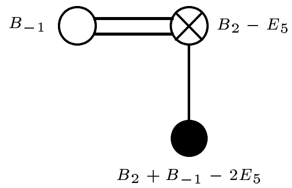


#### 6.4.5 $n = 3$

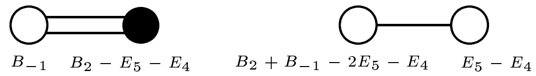
$D = 5$




$D = 4$

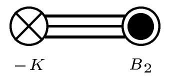



$D = 3$

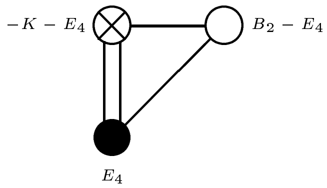



#### 6.4.6 $n = 2$

$D = 4$



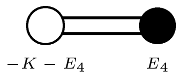

$D = 3$



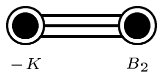

#### 6.4.7 $n = 1$

$D = 4$


$D = 3$

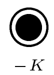



or




#### 6.4.8 $n = 0$

$D = 3$




Note that in all the diagrams of rank one where the simple root is $-K$ one is actually considering pure gravity. Again there are some purely bosonic examples of Borcherds algebras.

## 7. Conclusion

In [28, 29, 30, 31], it was noticed that modular integrals associated with one-loop calculations of compactified heterotic string theory can be interpreted as the Weyl denominator formula of a Borcherds (super)-algebra. We recall that the Weyl group is by definition generated by symmetries with respect to real roots and this denominator [10] can be written in two equivalent ways as:

$$\frac{\prod_{\alpha\ \mathrm{even}}(1-\alpha)^{\mathrm{mult}(\alpha)}}{\prod_{\alpha\ \mathrm{odd}}(1+\alpha)^{\mathrm{mult}(\alpha)}} = e^{\rho}\sum_{w\in W}\det(w)\,w(e^{-\rho}\sum_{\mu}\epsilon(\mu)e^{\mu}) \tag{7.1}$$

where the products on the left handside run over positive roots $\alpha$, respectively bosonic and fermionic, with multiplicities mult$(\alpha)$, $\rho$ is the Weyl vector, defined by $(\rho,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)$, $W$ is the Weyl group, the sum on the right handside runs over $\mu$ itself sum of mutually orthogonal imaginary simple roots

$$\mu = \sum_j \alpha_{i_j\mathrm{even}} + \sum_k l_{i_k}\alpha_{i_k\mathrm{odd}}$$

for positive integers $l_{i_k}$ multiplying the odd imaginary roots restricted by the condition that $l_{i_k}\geq 2$ is allowed only if $\alpha_{i_k\mathrm{odd}}$ is isotropic i.e. $(\alpha_{i_k\mathrm{odd}},\alpha_{i_k\mathrm{odd}})=0$ and $\epsilon(\mu)=(-1)^{\mathrm{ht}(\mu)}$.

One can implement the defining relations of Borcherds algebras most efficiently by using recursively the above formula. We only need a finite number of checks and this is a finite process. The singularities of the denominator function, studied by [32], had been interpreted in [28] as the enhanced symmetry points in the even self-dual Narain lattice. We plan to return to this problem.

In order to summarize we may say that we have considerably enlarged the finite dimensional duality superalgebras that themselves were a generalization of the U-duality symmetries. We may now return to the many open questions about the structure of Einstein, supergravity and superstring equations with hope of even more stunning beauty. Clearly the real algebraic geometry will be important, the fermionic extensions will be closely related to it. We may note that the positive degree obstacle from differential forms has been removed so we may go beyond Borel subgroups again and deal with the spacetime fermions. The symmetry of the Magic triangle fits in a considerable body of dualities. Of course the twistor-like construction of spacetime as a derived object is an open problem but one can see where to look for it already.

## Acknowledgments

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